# On satisfiable Tseitin formulas, branching programs and tree-width

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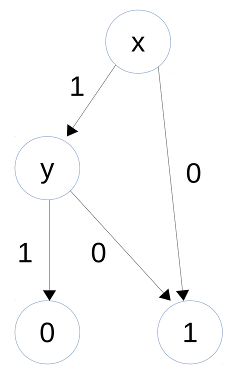
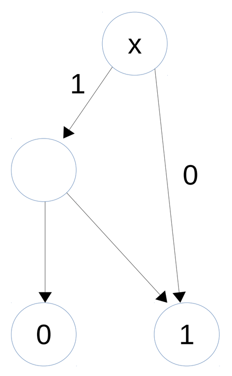
## Branching program




- Acyclic directed graph
- Two sinks: 1-sink and 0-sink
- All vertices are labeled by variables
- Value: sink label at the end of the path that corr. to the subst.
- Size = number of nodes

**Nondeterministic branching program**




- Has nodes without labels
- Value equals one if there **exists** a path from source to 1-sink
- Size = number of labeled nodes

**Read-k branching program (k-BP)**

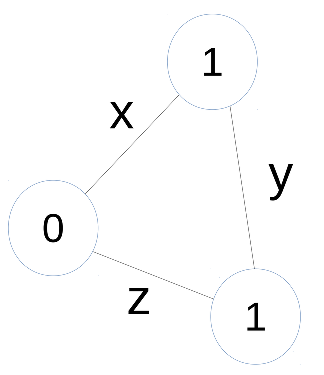
- Every path has only ***k* occurrences** of each variable

**Ordered binary decision diagram (OBDD)**

- **1-BP** in which all variables on all paths occur in the same order

## Tseitin formula

- Defined on a graph
- Variables on edges
- 0-1 values in vertices
- ***True*** iff for every vertex the sum of values on its edges equals its label




$$\begin{cases} x+y=1 \\ x+z=0 \\ y+z=1 \end{cases}$$

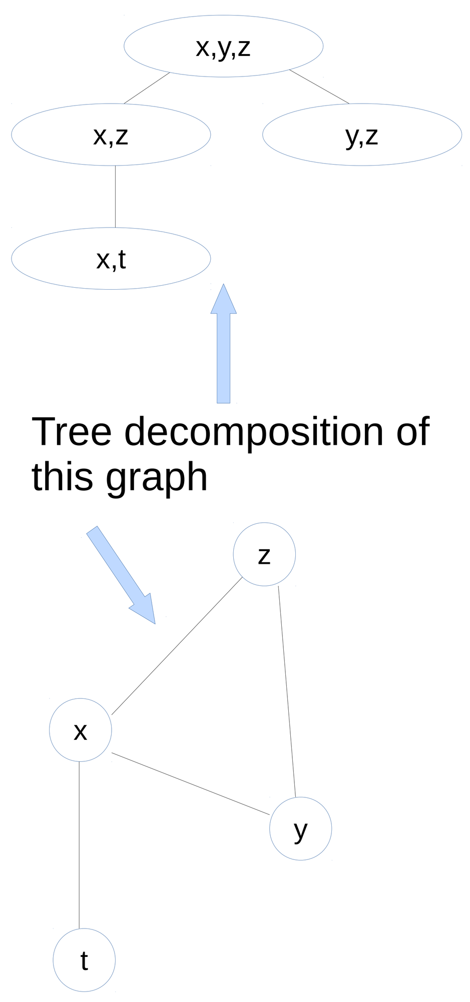
**Criterion of satisfiability:**

Tseitin formula is satisfiable iff for every connected component the sum of labels of vertices is even.

## Tree and path decomposition

**Tree decomposition** of a graph:

1. Vertices – “bags” are sets of vertices of $G=(V,E)$
2. If $(a,b)\in E$ then there is a bag with $a$ and $b$
3. All bags with the same vertex form a tree




**Path decomposition** of a graph:

1,2 are the same

3. All bags with the same vertex form a path

**Width** of a decomposition is the size of a maximal bag in it minus 1.

**Tree- (path-) width of a graph** is the minimal width among all its tree (path) decompositions.

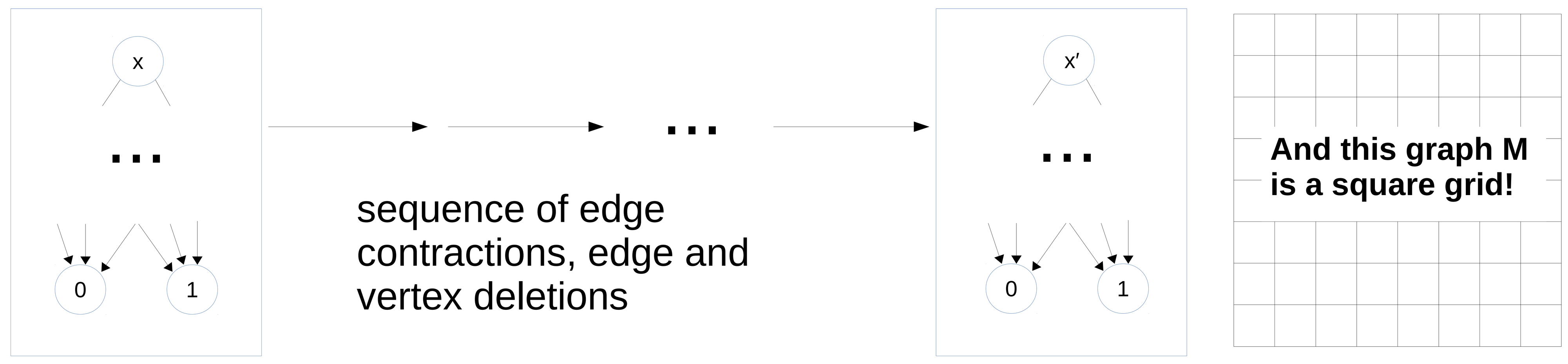
## Main result:

**1-NBP** for satisfiable Tseitin formula on a graph $G$ has size $2^{\Omega(t^\delta)}$, where $t$ is the tree-width of $G$, $\delta$ is a constant.

To prove this lower bound we use the **Excluded Grid Theorem** by Robertson and Seymour:

Every graph $G$ of a tree-width $t$ has a grid minor of size $t^\delta$, where $\delta$ is a constant.

**Minor** of $G$ is a subgraph of $G$ that can be obtained from $G$ by a sequence of edge contractions, edge and vertex deletions.




**1-NBP** for a Tseitin formula on graph $G$

**1-NBP** for Tseitin formula on minor $M$ of graph $G$

**Observation:** these operations lead to a smaller **1-NBP**

**Main lemma: 1-NBP** for a satisfiable Tseitin formula on an $n\times n$ grid graph has size $2^{\Omega(n)}$

To prove the Main lemma we used our previous result:

**Theorem: 1-NBP** for a satisfiable Tseitin formula on a graph $G$ has at least $2^{|V|-k_G(l)-k_G(|E|-l)-l+1}$ nodes, where the value $k_G(l)$ denotes the maximal number of connected components that can be obtained from $G$ by deleting $l$ edges.

Using this lemma for $n\times n$ grid graph we need to calculate values of $k_G(l)$ and $k_G(|E|-l)$ for some $l$ in a way that the difference of their sum from $|V|$ will be linear in $n$.

In particular we choose **$l = |E| / 2$**

We also obtained an **upper bound** for **OBDD** for satisfiable Tseitin formulas:

**Theorem:** a satisfiable Tseitin formula on graph $G$ can be computed by an **OBDD** of size $O(m2^{p+1})$, where $m$ is the number of edges and $p$ is the path-width of $G$.

We build **OBDD** layer-by-layer, each layer corresponds to one edge. Each edge corresponds to the first bag where it appears in the decomposition.

For every edge and its bag we add at most $2^{p+1}$ vertices that correspond to all possible values of parity in vertices from that bag.