MCSP is Hard for Read-Once Nondeterministic Branching Programs

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Outline

- Minimum Circuit Size Problem
- Branching Programs
- Our result: every 1-NBP computing MCSP has superpolynomial size
- Technique

Minimum Circuit Size Problem

Input:

- truth table of a Boolean function $f: \{0,1\}^n \to \{0,1\}$

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Truth table of f of length $N = 2^n$

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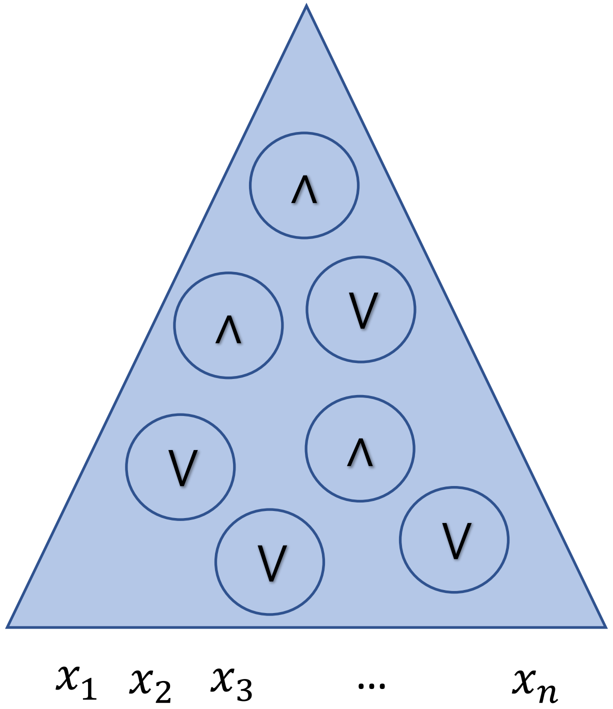
Output:

yes, if f can be computed by a circuit of size at most s



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- Complexity of MCSP in restricted classes is important too:
  If MCSP cannot be computed by
    - a branching program of size $N^{2.01}$
    - formula of size $N^{3.01}$
    - circuit of size $N^{1.01}$

  Then NP $\not\subset$ C-SIZE[n^k] for all k [Chen, Jin, Williams, 2019]

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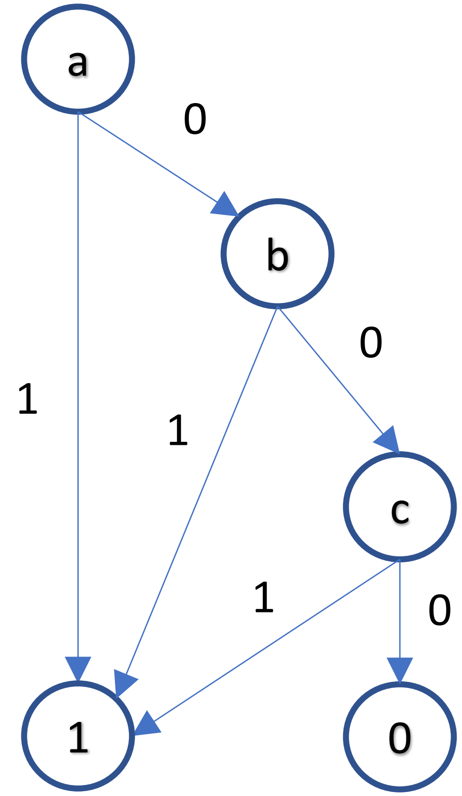
- 1-coNBP(MCSP)= $2^{\Omega(N)}$ [Cheraghchi, Hirahara, Myrisiotis, Yoshida, 2019]

Branching program

- BP is a way to represent Boolean function:
  - directed graph without cycles
  - one source
  - two sinks: labeled with 0 and 1
  - all other vertices labeled with variables
  - values of variables on edges
- Size of a BP is a number of vertices



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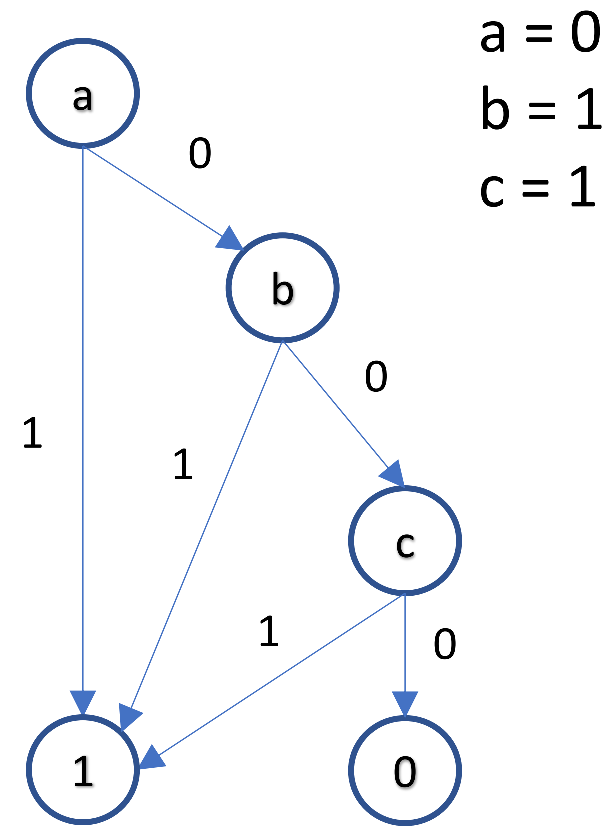

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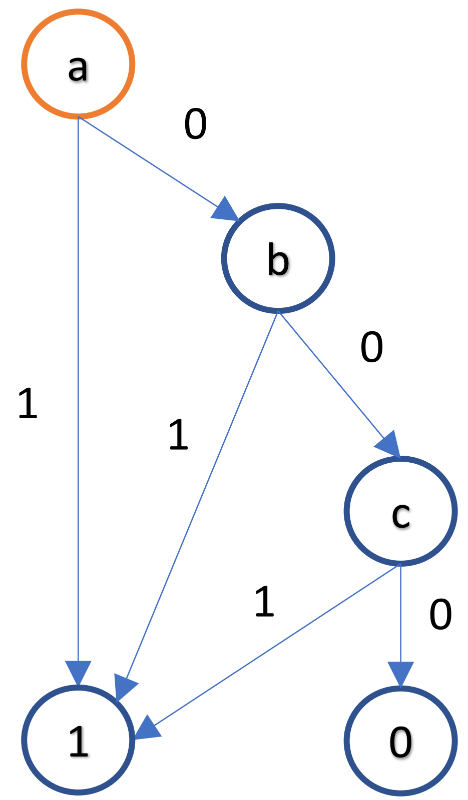

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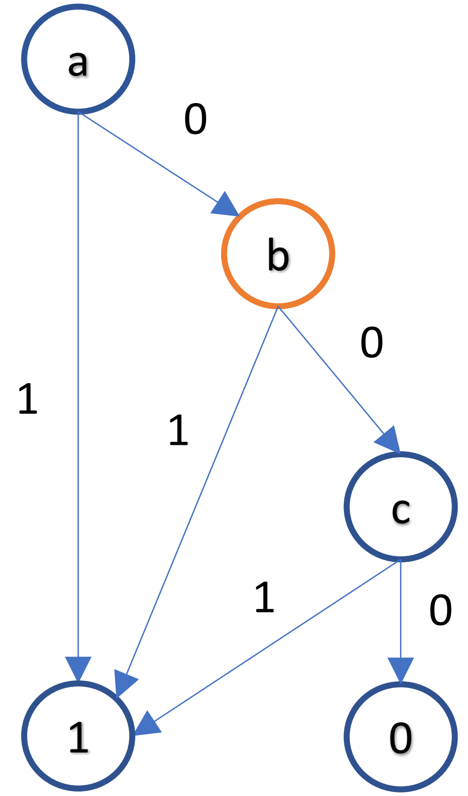

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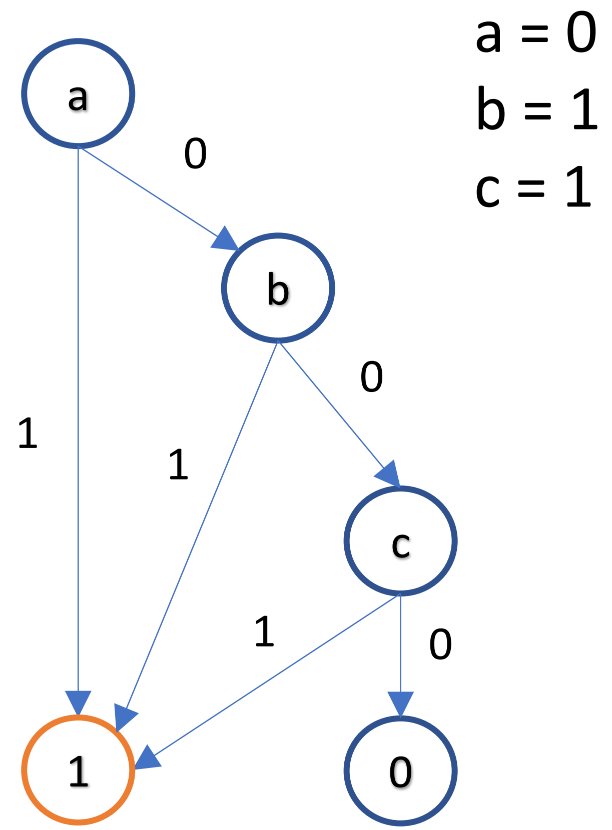

Non-deterministic branching program

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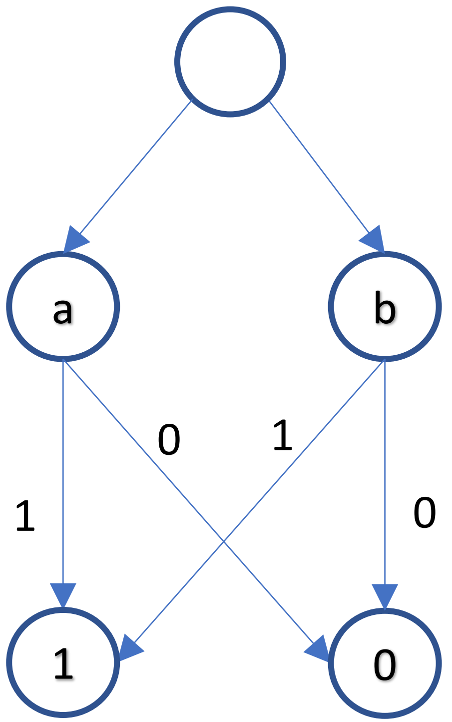
  - non-deterministic nodes are unlabeled
  - the value equals 1 $\Leftrightarrow$ exists a path to 1-sink



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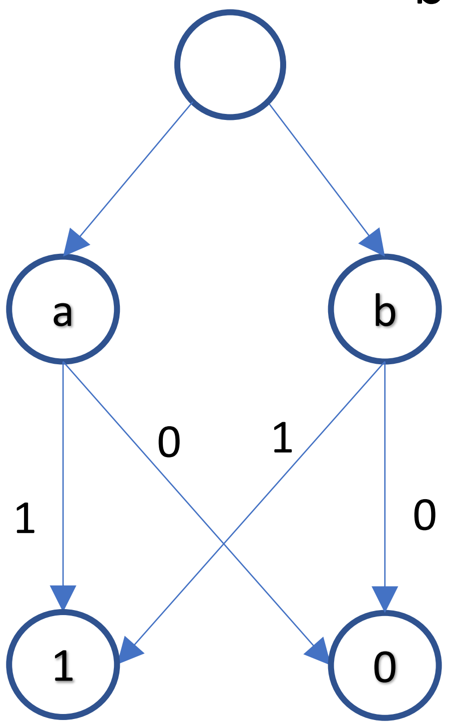

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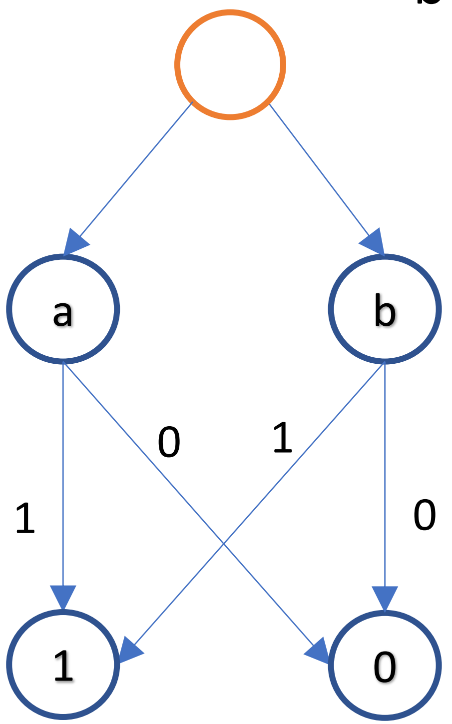

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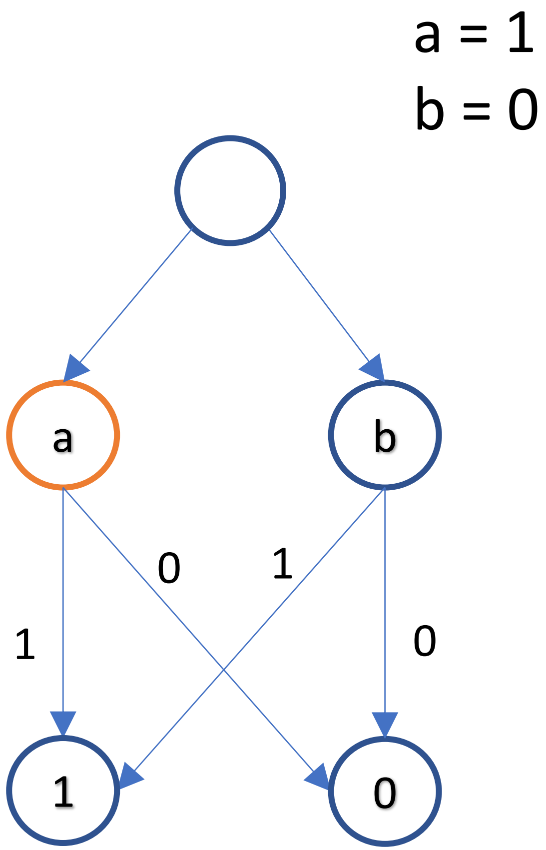

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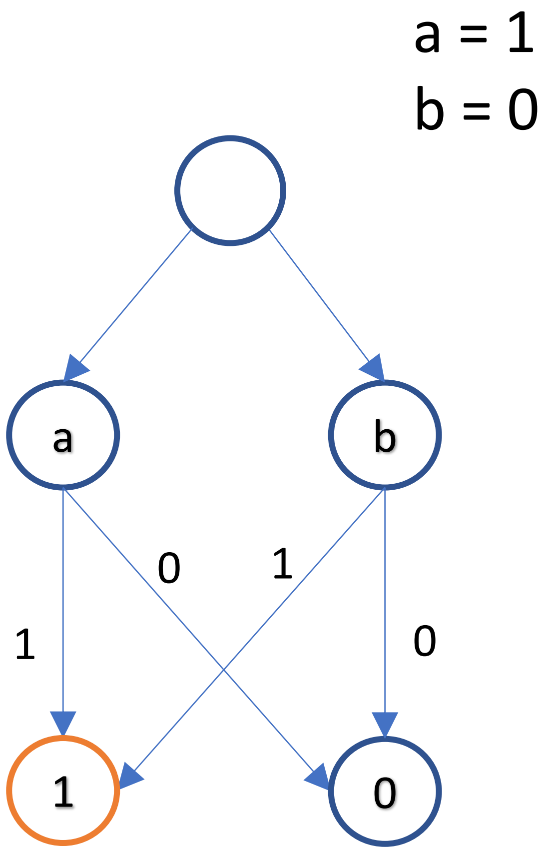

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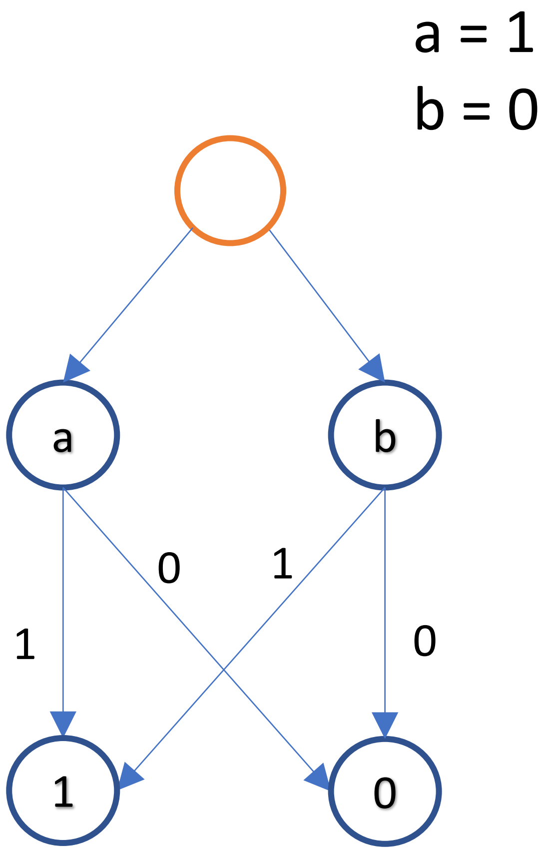

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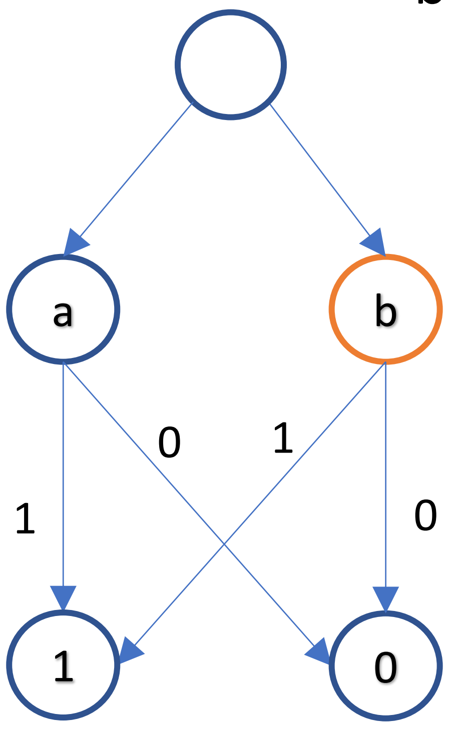

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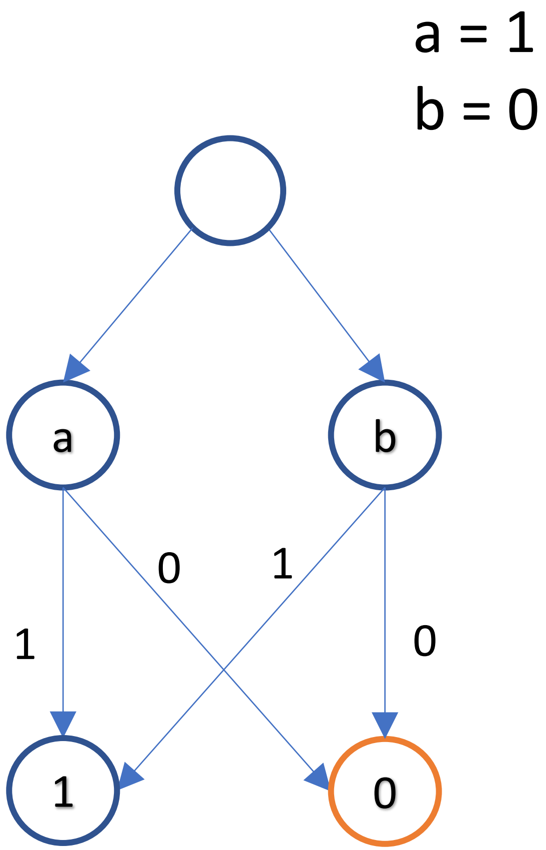
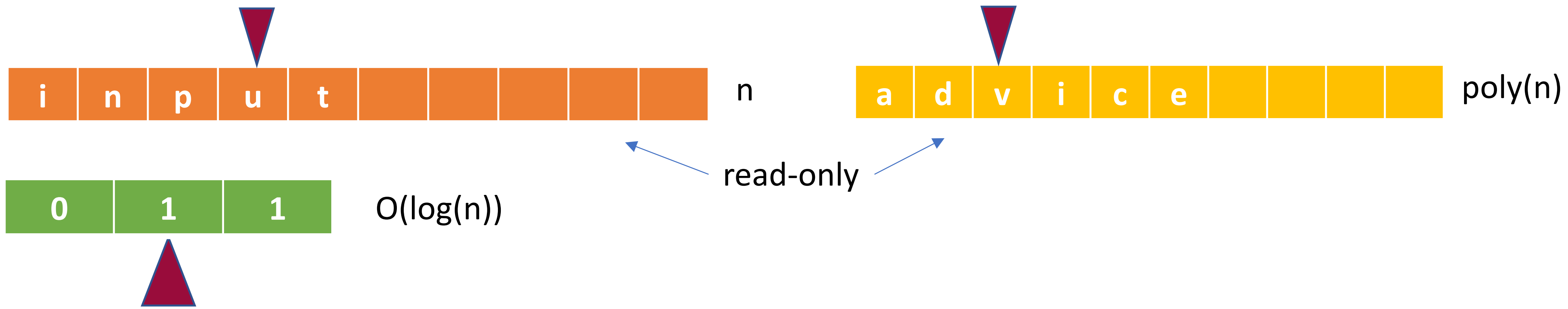


Complexity class with logarithmic space



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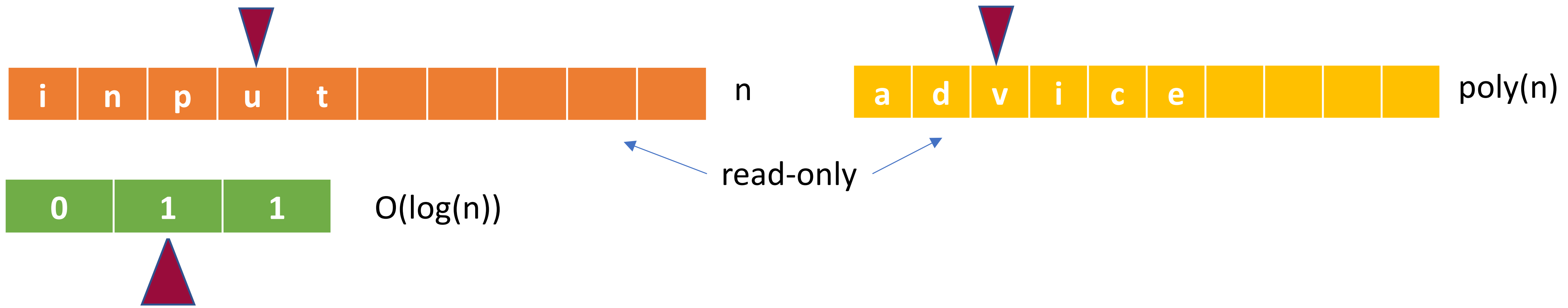

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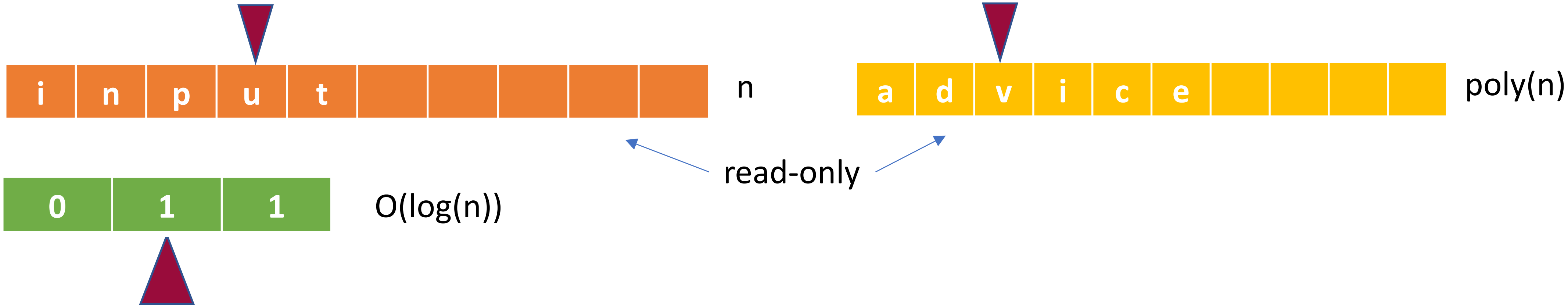

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- NBP corresponds to NL/poly

Best lower bounds for branching programs

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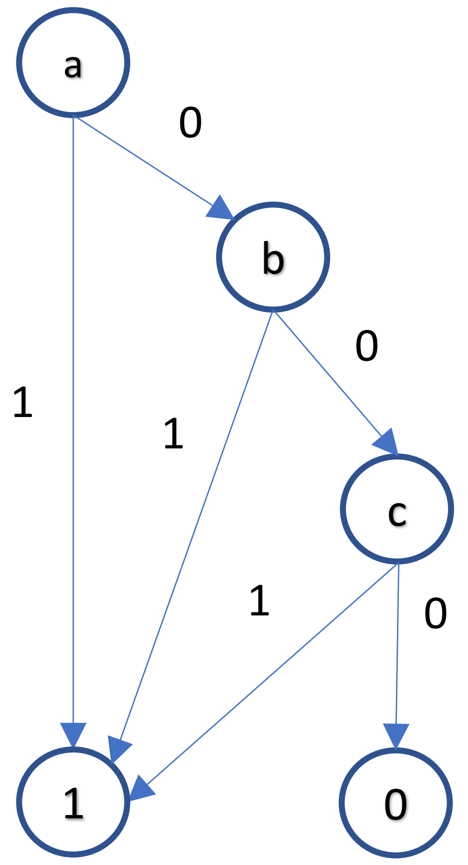
- The best lower bound: BP(ED)=$\Omega\left(\frac{n^2}{\log^2 n}\right)$ [Nechiporuk, 1966]
- Recent results:
  - BP(MCSP)=$\widetilde{\Omega}(N^2)$ [Cheraghchi, Kabanets, Lu, Myrisiotis, 2019]
  - Barrier on proving better than $\widetilde{\Omega}(N^2)$ for MCSP [Chen, Jin, Williams, 2019]

Read-once branching programs

1-BP (1-NBP) if for every path every variable occurs no more than 1 time



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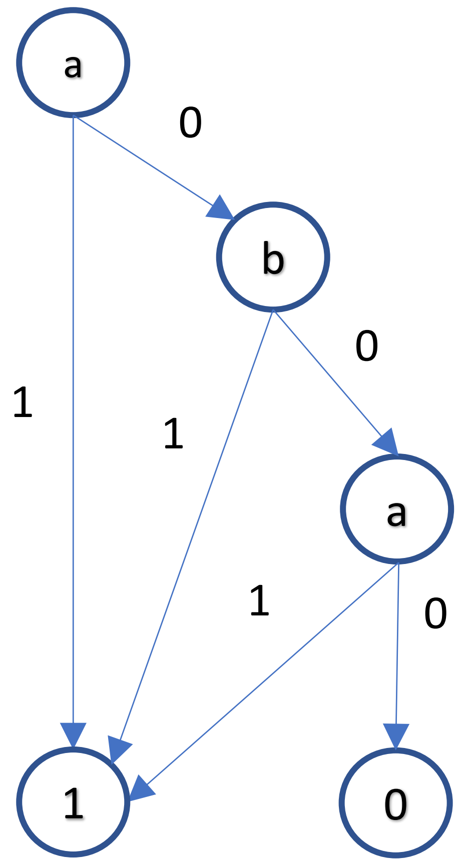

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MCSP naturally a nondeterministic problem, so it is harder to prove a lower bound against NBP

Main result

Theorem: size of 1-NBP computing MCSP is $N^{\Omega(\log\log N)}$

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(n x n)-Bipartite Independent Set Problem

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(n x n)-Bipartite Independent Set Problem $\longrightarrow$ MCSP*

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(n x n)-Bipartite Independent Set Problem

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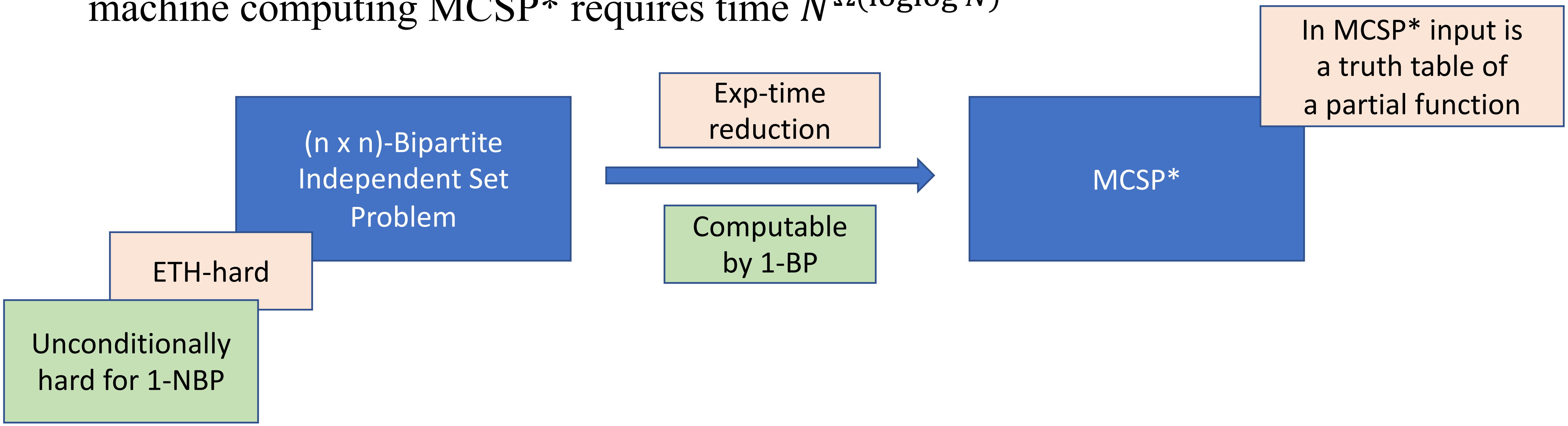

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Exp-time reduction

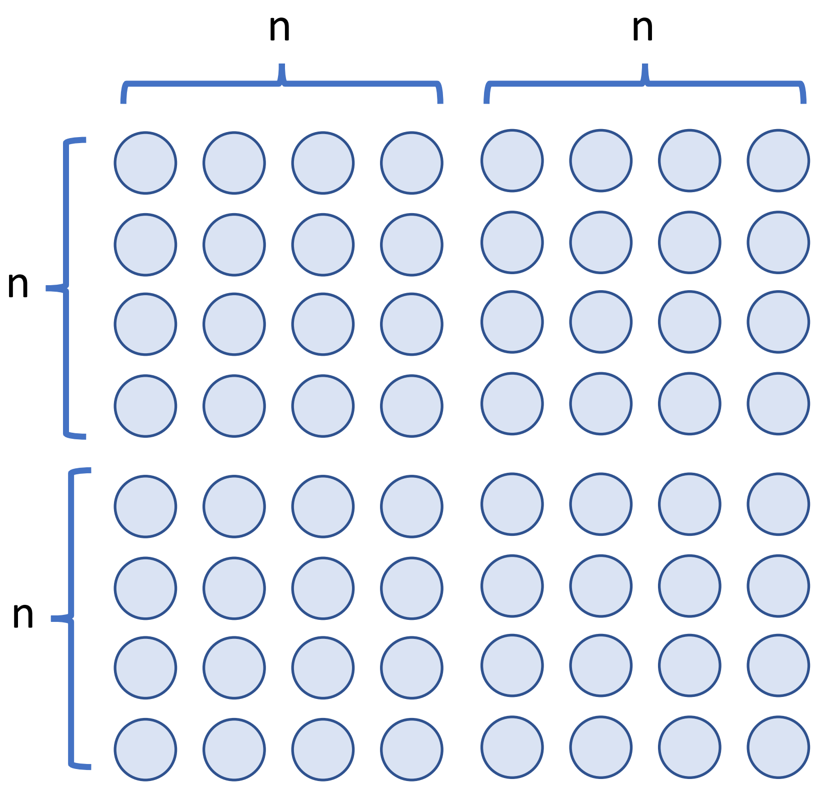
Computable by 1-BP

MCSP*

MCSP

Have the same 1-NBP complexity

(n x n)-Bipartite Permutation Independent Set (BPIS)



- Graph with 2n x 2n vertices,
- Edges exist only between vertices from two quadrants
- Need to find exactly one vertex from every row, and exactly one vertex from every column, such that
  - These vertices are from the two quadrants
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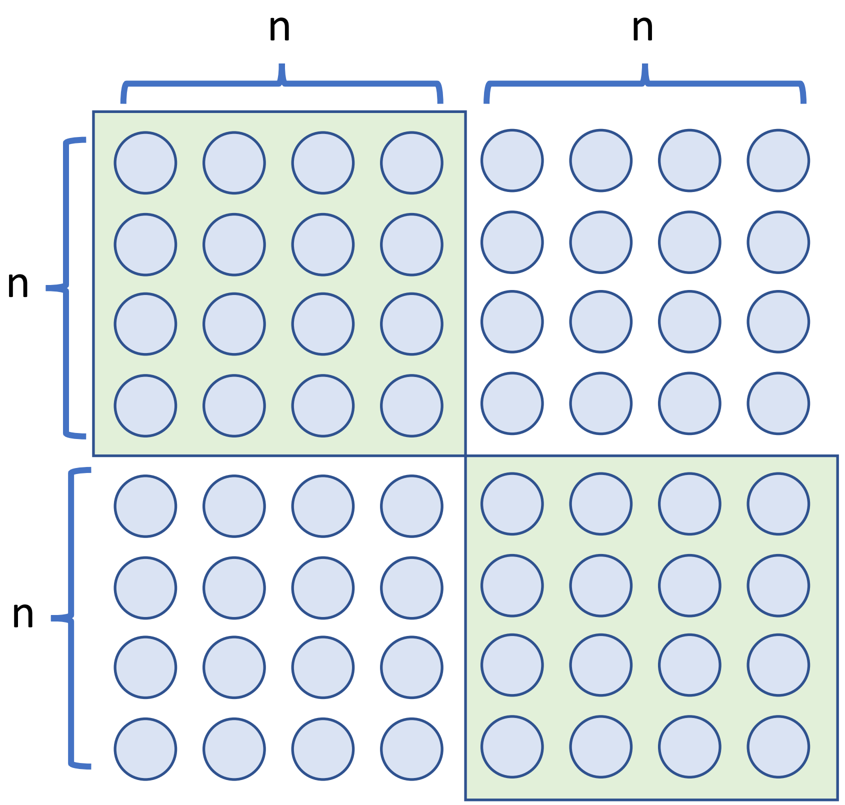

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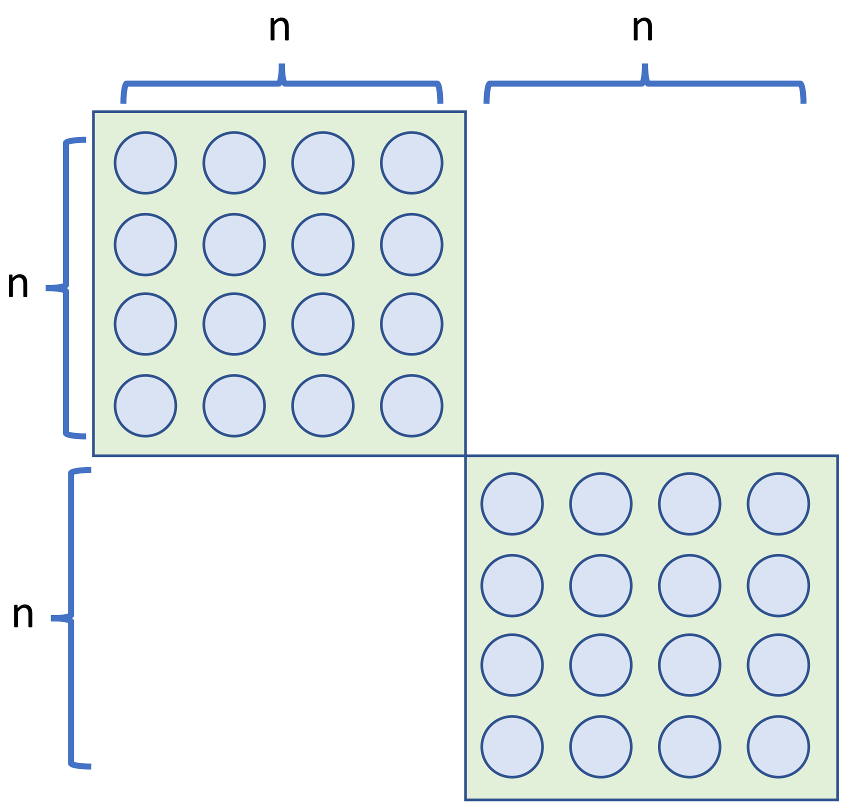

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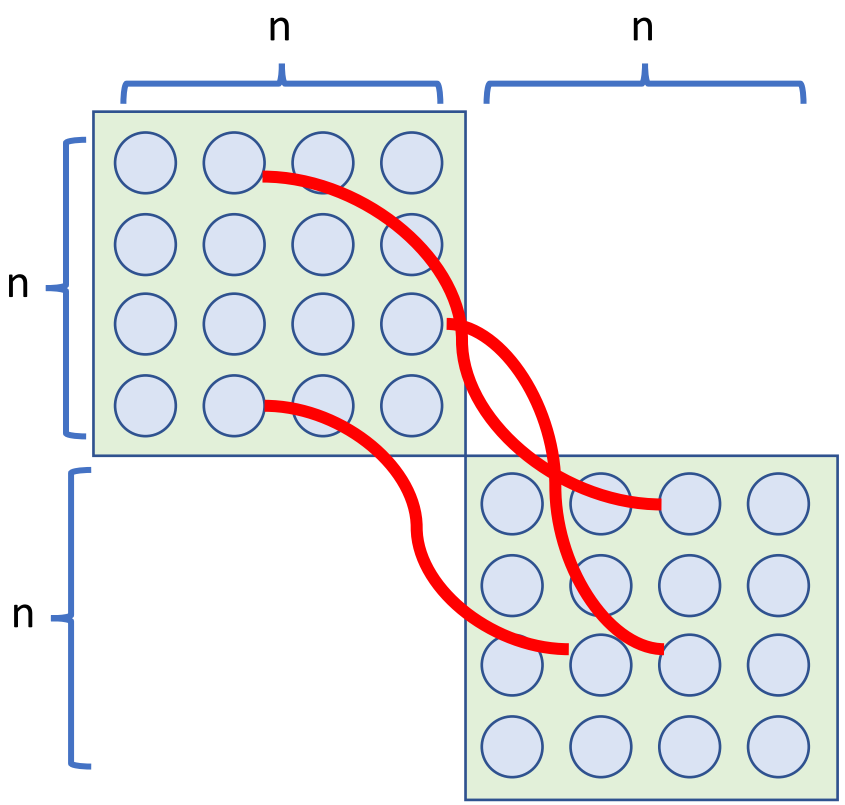

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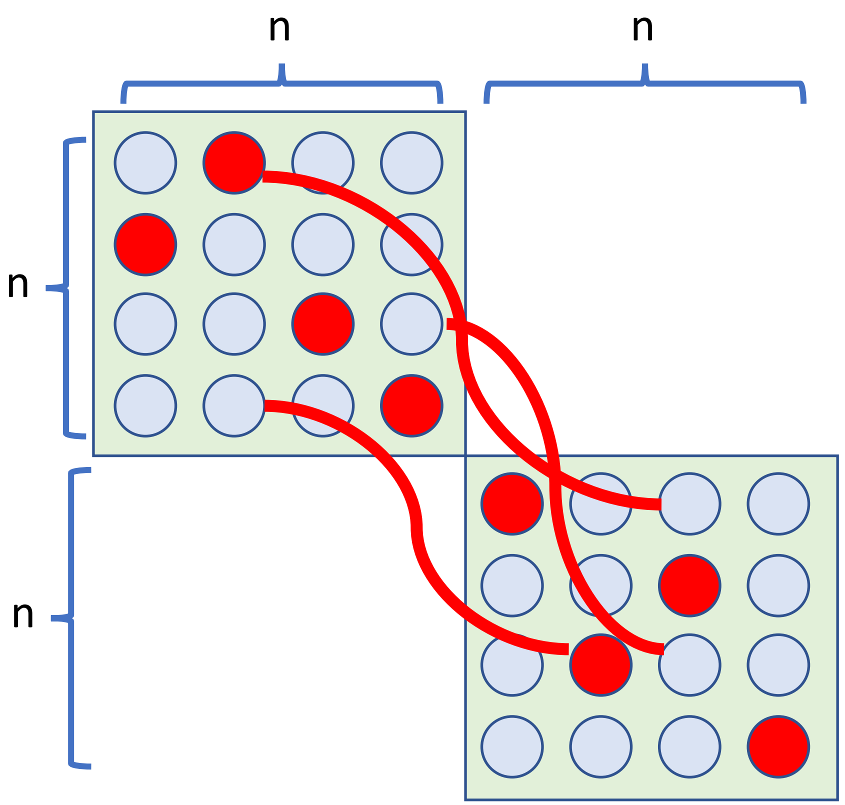

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(n x n)-BPIS is hard for 1-NBP

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Idea of the proof:

- Show that the minimum 1-NBP for Bipartite Permutation Independent Set has the same size as the minimum 1-NBP for Bipartite Permutation Clique

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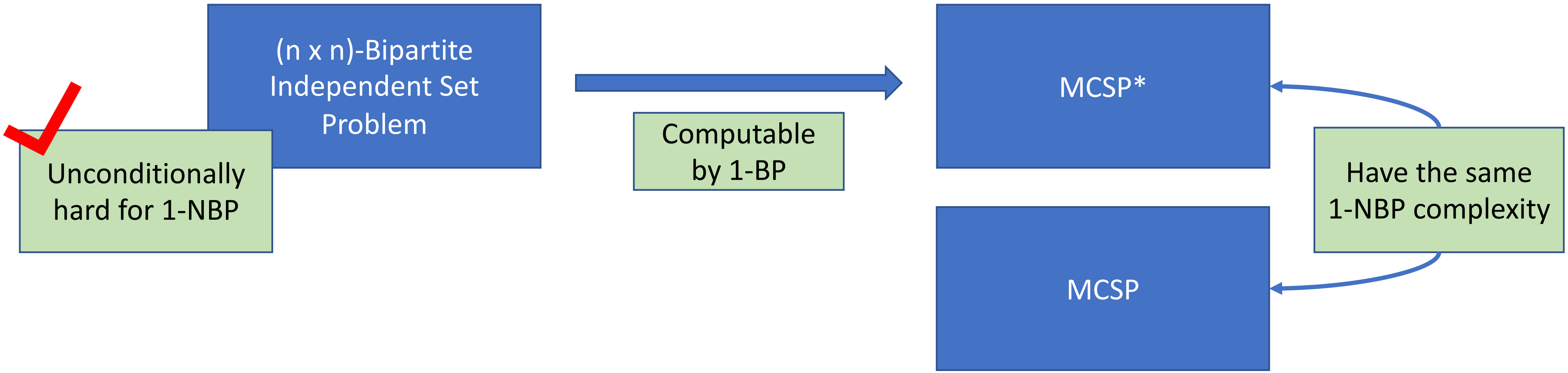
- Adapt the proof of the lower bound on 1-NBP for CLIQUE_ONLY to get a lower bound on BPC

Progress so far



1-NBP for MCSP* can be transformed to 1-NBP for BPIS

$$\gamma(x,y,z) = \begin{cases} \bigvee_{i\in[2n]}(y_i \wedge z_i) & \text{, if } x = 0^{2n} \\ \bigvee_{i\in[2n]} z_i & \text{, if } x = 1^{2n} \\ \bigvee_{i\in[2n]}(x_i \vee y_i) & \text{, if } z = 1^{2n} \\ 0 & \text{, if } z = 0^{2n} \\ \mathsf{OR}_n(x_1,\dots,x_n) & \text{, if } z = 1^n0^n \text{ and } y = 0^{2n} \\ \mathsf{OR}_n(x_{n+1},\dots,x_{2n}) & \text{, if } z = 0^n1^n \text{ and } y = 0^{2n} \\ 1 & \text{, if } \exists\ ((j,k),(j',k')) \in E \text{ such that } (x,y,z) = (\overline{e_k e_{k'}}, 0^{2n}, e_j e_{j'}) \\ \star & \text{, otherwise} \end{cases}$$

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Only these bits of the truth table depend on the input bits of BPIS

1-NBP for MCSP* can be transformed to 1-NBP for BPIS

$$\gamma(x,y,z) = \begin{cases} \bigvee_{i\in[2n]}(y_i \wedge z_i) & \text{, if } x = 0^{2n} \\ \bigvee_{i\in[2n]} z_i & \text{, if } x = 1^{2n} \\ \bigvee_{i\in[2n]}(x_i \vee y_i) & \text{, if } z = 1^{2n} \\ 0 & \text{, if } z = 0^{2n} \\ \mathsf{OR}_n(x_1,\ldots,x_n) & \text{, if } z = 1^n0^n \text{ and } y = 0^{2n} \\ \mathsf{OR}_n(x_{n+1},\ldots,x_{2n}) & \text{, if } z = 0^n1^n \text{ and } y = 0^{2n} \\ 1 & \text{, if } \exists\, ((j,k),(j',k')) \in E \text{ such that } (x,y,z) = (\overline{e_k e_{k'}}, 0^{2n}, e_j e_{j'}) \\ \star & \text{, otherwise} \end{cases}$$

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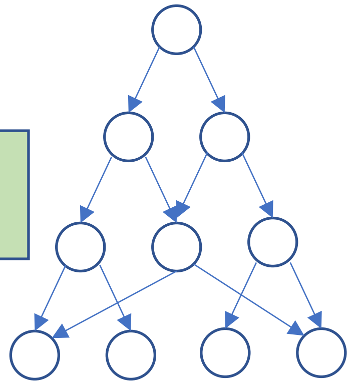
1-NBP for MCSP*



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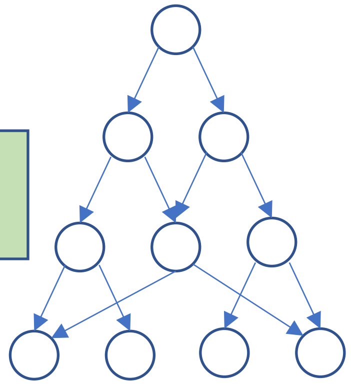
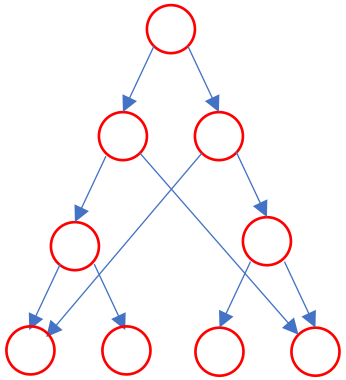


Substitute bits of the truth table of γ that do not depend on BPIS' input



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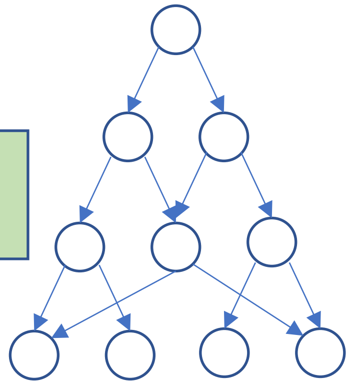

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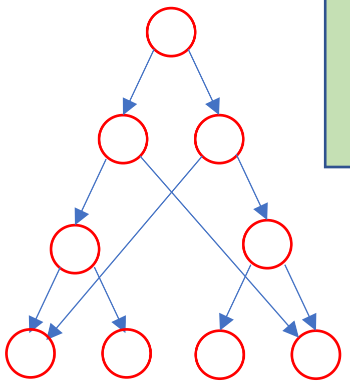
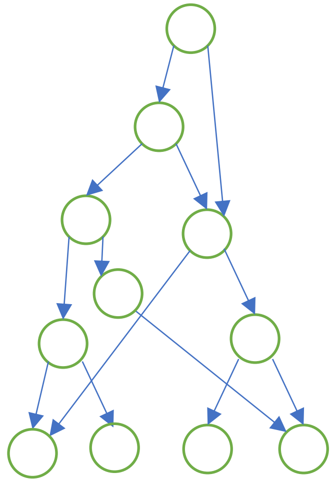
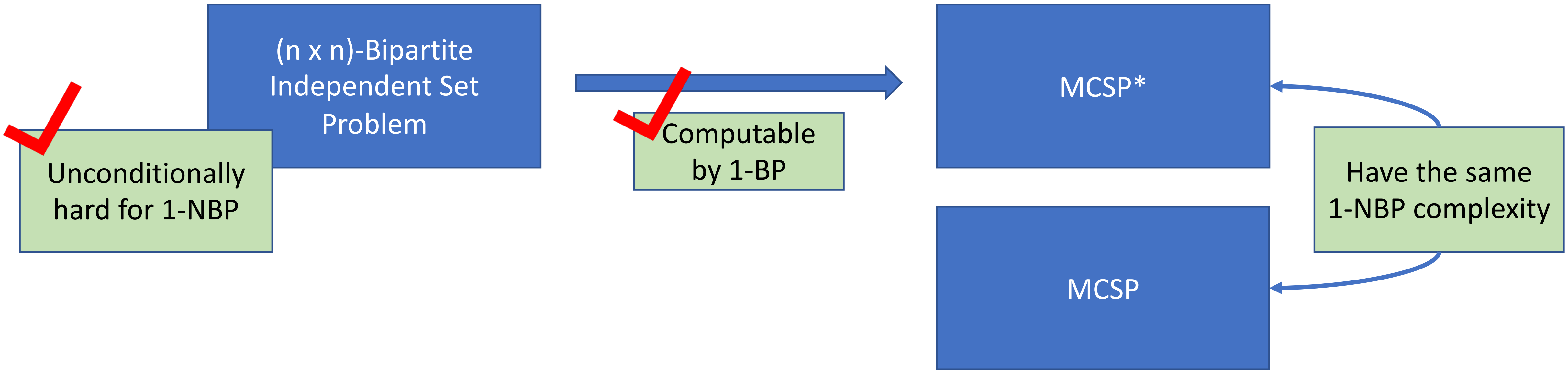


Substitute 1-BPs that computes dependency on the edges of BPIS



Almost finished



MCSP* and MCSP have the same 1-NBP complexity

Lemma: the size of the minimal 1-NBP computing MCSP* equals the size of the minimal 1–NBP computing MCSP

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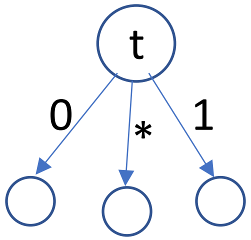

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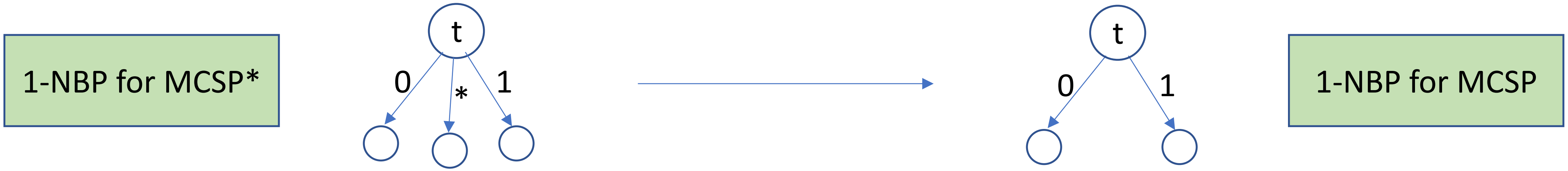

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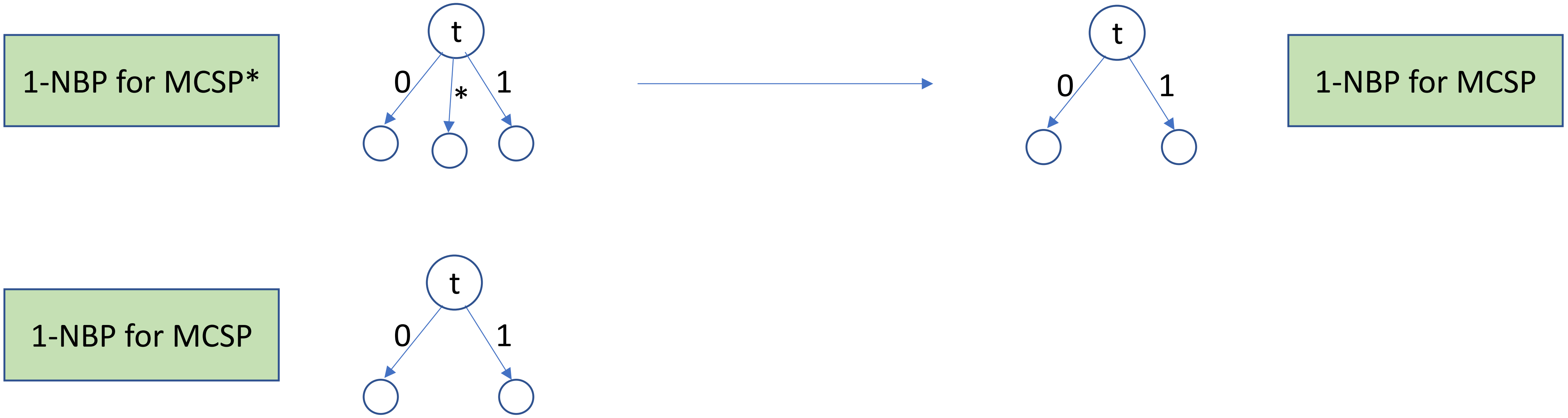

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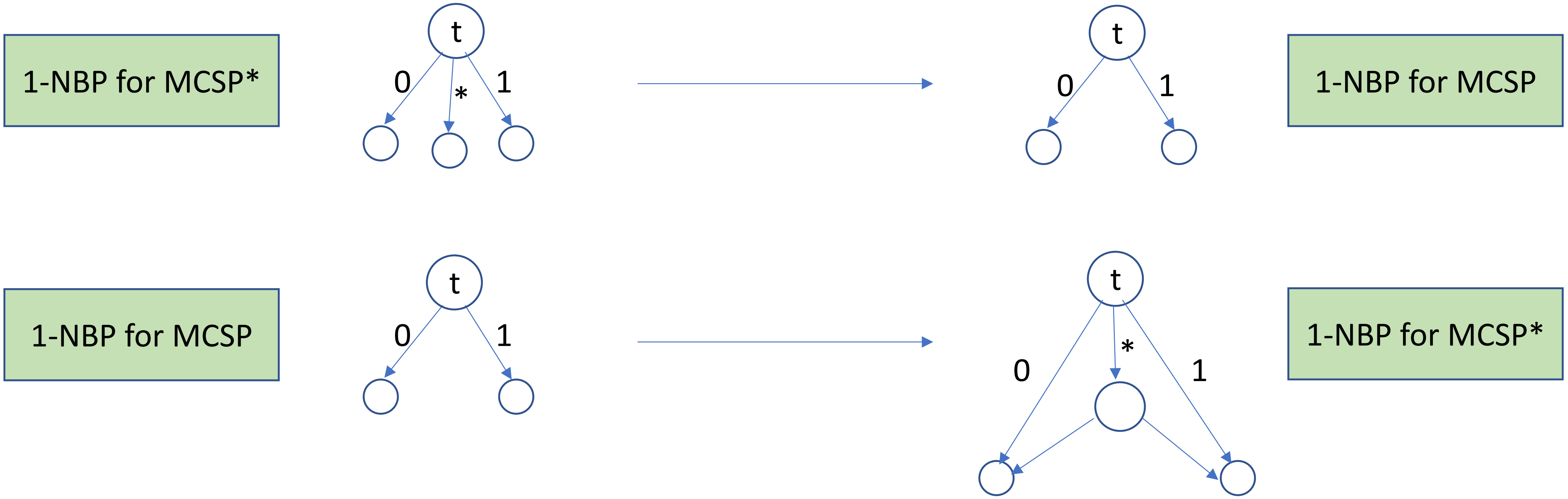
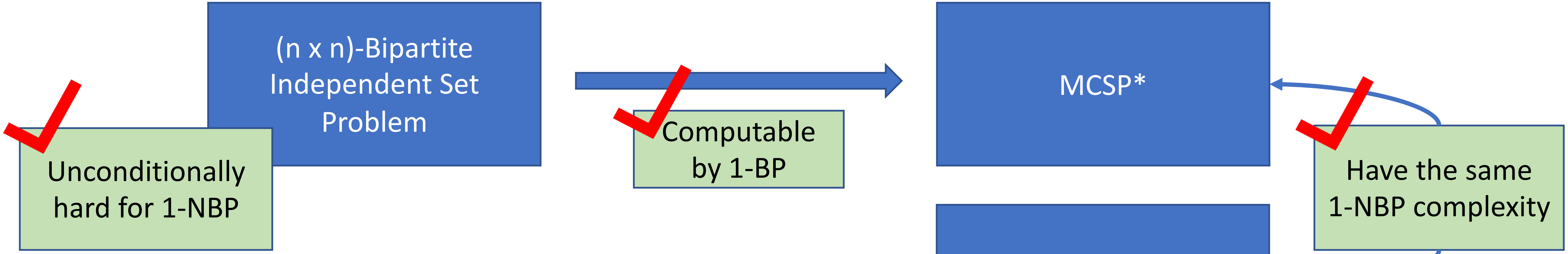


Putting all together



If MCSP could be computed by a small 1-NBP, then (n x n)-BPIS could be computed by a 1-NBP of even smaller size.
That leads to a contradiction.

Upper bound

Lemma: MCSP on an input of length 2^n with a size parameter s can be computed by a 1-NBP of size $2^n 2^{O(s \log s)}$

Upper bound

Simple guess
and check strategy

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Corollary: our $2^{\Omega(n \log n)}$ lower bound is tight for inputs with a linear size parameter

Open questions

- Show tight lower bound for MCSP with higher size parameters
  - The same technique cannot work, as we cannot construct a truth table of a function with higher than linear circuit complexity

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- Show tight lower bound for MCSP with higher size parameters
  - The same technique cannot work, as we cannot construct a truth table of a function with higher than linear circuit complexity
- Extend this result to other models of computations
  - For any model in which (n x n)-BPIS is hard and the reduction to the truth table is efficiently computable the same size lower bound will hold

Partial Minimum Circuit Size Problem

Input:

- truth table of a partial Boolean function
  $f: \{0,1\}^n \to \{0,1,*\}$
- size parameter s

1	*	*	1	*	1	1	0	…	1

Truth table of f of length $N = 2^n$

Output:

yes, if exists a total function g that is consistent with f and can be computed by a circuit of size at most s